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Multiplication of integral octonions

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The integral subsets of octonions are an analog of integers in real numbers and related to many interesting topics in geometry and physics via E_8-lattices. In this paper, we study the properties of the multiplication of the integral subsets of octonions by studying configuration of Fano plane via blocks and operations on them. And we show that the integral subsets are integral indeed by introducing new and elementary methods.

Keywords: Integral quaternion; integral octonion.

Mathematics Subject Classification: 17A75, 17A99

1. Introduction

The octonions are normed division algebras whose classification consists of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. The octonions are a very complicated algebra since its product is neither commutative nor associative. The subtle product of $\mathbb{O}$ is the main source of the most complex issues in the exceptional groups including G_2 and E_8 which are related

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to certain types of symmetries which drive attraction to physics and geometry. In fact, one can find numerous studies utilizing the octonions in physics including the Great Unification Theory or the Theory of Everything. Very often, the E_8-lattice (also called the Gosset lattice), which is a unimodular root lattice of rank 8, appears as a key player to convey the above symmetries on behalf of the octonions.

On the other hand, the E_8-lattice can be considered a subset in $\mathbb{O}$ which is called an integral octonion, an analog of integers in the real numbers. In particular, Coxeter [3] worked on the integral octonions (also called integral Cayley numbers) to study the Gosset polytope 4_{21} which is an eight-dimensional uniform polytope with E_8-symmetry. Koca *et al.* [4–6] studied the symmetries given by the integral octonions and the pure integral octonions, and applied the studies to mathematical physics. Recently, Conway and Smith explained the integral octonions and their relationship to the E_8-lattice in [2].

Even though the integral octonion appears as one of the fundamental figures in many interesting research topics, the nature of the integral octonion given by the multiplication of an integral subset of a normed algebra is far beyond understood. From this motivation, we study the integral subsets of normed algebras. In particular, we study the multiplication in $\mathbb{O}$ according to integrality and produce new and elementary proof of closedness of certain subsets for the multiplication so that they are indeed integral octonion subsets. Therein, we also provide rather explicit studies on the multiplications of the integral subsets which bring many subtle puzzles.

In Sec. 2, according to [3], we introduce the definition of integral subsets of normed division algebras and recall well-known examples of them. In Sec. 3, we introduce octonions along the Fano plane. According to the configuration of Fano plane we define normal blocks and conormal blocks which play key roles to study the multiplications of octonions, and introduce *swaps* which are operations on the set $\mathcal{B}$ of those blocks. Well known as Kirmse's mistake (see [3]), the subset $\mathbb{O}_Z$ defined as

$$\mathbb{O}_Z := \mathrm{span}_{\mathbb{Z}} \left\{ \frac{1}{2}(\pm e_a \pm e_b \pm e_c \pm e_d) \in \mathbb{O} \,|\, \{a, b, c, d\} \ \in \mathcal{B} \right\}$$

is not closed under multiplication. Therefore, we twist the blocks $\mathcal{B}$ by *swaps* $\sigma_{(i,j)}$ to define new subset $\mathbb{O}_Z(i, j)$ as

$$\mathbb{O}_Z(i, j) := \mathrm{span}_{\mathbb{Z}} \left\{ \frac{1}{2}(\pm e_a \pm e_b \pm e_c \pm e_d) \in \mathbb{O} \,|\, \{a, b, c, d\} \ \in \sigma_{(i,j)}(\mathcal{B}) \right\}$$

and show $\mathbb{O}_Z(i, j)$ is a integral subset $\mathbb{O}$. Here we provide a new proof by introducing elementary and self-contained methods to show the closedness for the multiplication. We also list the generators of Weyl groups for the Dynkin diagram which identify the integral subset $\mathbb{O}_Z(0, 7)$ as an E_8-lattice in $\mathbb{O}$.

2. Integral Subsets of Algebras

In this section, we introduce the definition of integral subsets of algebras in the Coxeter's work [3] via the modern treatment of the normed division algebras in [1, 2].

Let $\mathcal{A}$ be an algebra which is a finite-dimensional vector space over $\mathbb{R}$ equipped with a multiplication $\cdot$ and its unit element 1. Moreover the algebra is called a *normed* (*division*) *algebra* if the algebra is also a normed vector space with a norm $\| \ \|$ satisfying $\|a \cdot b\| = \|a\|\|b\|$ for all a and b in $\mathcal{A}$. Here each norm on the normed algebra gives a derived inner product defined by

$$2(a,b) := \{\|a+b\|^2 - \|a\|^2 - \|b\|^2\},$$

and each a in $\mathcal{A}$ satisfies a *rank equation*

$$a^2 - 2(a,1)a + \|a\|^2 = 0.$$

It is well known that the classification of the normed algebras consists of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

Definition 1. A subset S of a normed algebra $\mathcal{A}$ is called integral (or a set of integral elements) if it satisfies the following conditions:

(1) For each element S, the coefficients of the above rank equation are integers.
(2) The set S is closed under subtraction.
(3) The set S is closed under multiplication.
(4) $1 \in S$.
(5) S is a maximal subset in A with (1)–(4).

Example 2. When $\mathcal{A}$ is complex numbers, the subset

$$S_1 := \{a_0 + a_1 i \in \mathbb{C} \,|\, a_0, a_1 \in \mathbb{Z}\}$$

in $\mathbb{C}$ satisfies the above conditions. The elements of the subset are called the *Gaussian integers*. Here we check condition (5). Let S' be another subset in $\mathbb{C}$ where it satisfies above conditions (1)–(4) and $S' \supsetneq S_1$. Then there exist w in $S' - S_1$ and z in $S_1 \subset S'$ where the distance between z and w is less than 1, namely, $\|w - z\| < 1$. But since S' satisfies conditions (1) and (2), $w - z \in S'$ and $\|w-z\|^2 \in \mathbb{Z}$. This gives a contradiction to the existence of S'. Thus S_1 satisfies the condition (5).

Example 3. A subset $S_2 := \{a_0 + a_1 \frac{-1+\sqrt{-3}}{2} \in \mathbb{C} \,|\, a_0, a_1 \in \mathbb{Z}\}$ in $\mathbb{C}$ is an integral subset.

Example 4. The set of the quaternions $\mathbb{H}$ is a real four-dimensional vector space which is spanned by a basis $\{1, i, j, k\}$, and its normed algebra is given by an associative multiplication satisfying

$$i^2 = j^2 = k^2 = ijk = -1.$$

We consider a subset $\mathbb{H}_I$ in $\mathbb{H}$ consisting of the quaternions $a_0 1 + a_1 i + a_2 j + a_3 k$ whose coefficients a_0, a_1, a_2, a_3 are synchronically chosen from either $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, namely,

$$\mathbb{H}_I = \left\{ q \in \mathbb{H} \,\middle|\, \begin{array}{l} q = b_0 1 + b_1 i + b_2 j + b_3 k \text{ or} \\ b_0 1 + b_1 i + b_2 j + b_3 k + \frac{1}{2}(1 + i + j + k) \\ \quad \text{for } b_0, b_1, b_2, b_3 \in \mathbb{Z} \end{array} \right\}.$$

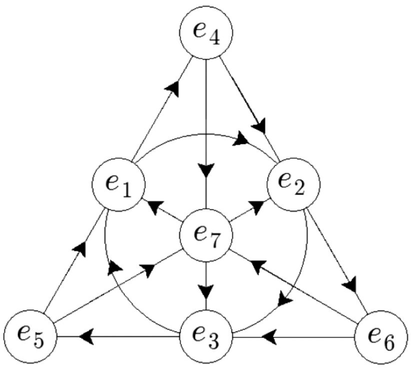
In fact, this subset $\mathbb{H}_I$ is integral and it is known as the Hurwitz integral quaternions. Checking conditions (1)–(4) can be done via rather simple computations and condition (5) can be checked by following the argument in Example 2.

3. Integral Octonions

The set of the octonions $\mathbb{O}$ is a real eight-dimensional vector space given as

$$\mathbb{O} := \{a_0 e_0 + a_1 e_1 + \cdots + a_7 e_7 \in \mathbb{O} \,|\, a_i \in \mathbb{R}\}$$

which has a multiplication defined by $e_0 = 1$, $e_1^2 = e_2^2 = \cdots = e_7^2 = -1$ and the following Fano plane:



Fano Plane

Here for any two different i and j in $\{1, 2, 3, 4, 5, 6, 7\}$ the multiplication of $e_i e_j$ is determined as $\pm e_k$ which is the third element in the unique line in the Fano plane containing e_i and e_j where $(+)$ sign is given if the order of the multiplication is matched with the direction of the arrow of the side and otherwise $(-)$ sign is given.

For any octonion $a = \sum_{i=0}^{7} a_i e_i$, the real part Re a and norm $\| \ \|$ are defined as Re $a := a_0 = (a, 1)$, $\|a\|^2 = \sum_{i=0}^{7} a_i^2$.

Just like the quaternions $\mathbb{H}$, the multiplication of the octonions $\mathbb{O}$ is not commutative, and furthermore, it is not associative. This lack of the associativity makes the research on the octonions $\mathbb{O}$ very difficult and complicated, but it is also the main source of anomalies in mathematical physics such as M-theory.

3.1. *Normal blocks and conormal blocks*

In this subsection, we introduce normal blocks and conormal blocks to the above Fano plane and an operation on the set of blocks called a *swap*. The blocks and swap will play key roles to study integral subset of octonions.

For each line in the Fano Plane and elements e_i, e_j, e_k in the line, we call the subscript subset $\{0, i, j, k\}$ of $I := \{0, 1, 2, 3, 4, 5, 6, 7\}$ and the corresponding complementary subset $I - \{0, i, j, k\}$ as a *normal block* (or *n-block*) and a *conormal block* (or *c-block*). The union of all n-blocks and c-blocks is denoted $\mathcal{B}$ and its element is referred as a block. In the following table, we list all n-blocks and c-blocks:

n-block	$\{0,1,2,3\}\{0,3,4,7\}\{0,2,5,7\}\{0,1,6,7\}\{0,1,4,5\}\{0,2,4,6\}\{0,3,5,6\}$
c-block	$\{4,5,6,7\}\{1,2,5,6\}\{1,3,4,6\}\{2,3,4,5\}\{2,3,6,7\}\{1,3,5,7\}\{1,2,4,7\}$

The blocks have the following interesting properties which are useful to study the integral subset in octonions.

Lemma 5. *For any two blocks B_1 and B_2 in $\mathcal{B}, |B_1 \cap B_2| = 2$ and $B_1 \triangle B_2 (:= B_1 \cup B_2 - B_1 \cap B_2)$ is also a block unless $B_1 \cap B_2 = \emptyset$ or $B_1 = B_2$.*

Proof. In the following, we only consider two blocks which are neither $B_1 \cap B_2 = \emptyset$ nor $B_1 = B_2$.

Case 1 (Both B_1 and B_2 are n-blocks). Since any two lines in the Fano plane share a vector, any two n-blocks share two subscripts in I including 0. Moreover, $B_1 \triangle B_2$ is a c-block because $|B_1 \triangle B_2| = 4$ and $B_1 \triangle B_2$ does not contain 0 and a line in the Fano plane.

Case 2 (Both B_1 and B_2 are c-blocks). Because $I - B_1$ and $I - B_2$ are n-blocks, we obtain $|B_1 \cup B_2| = 6$ from $2 = |(I - B_1) \cap (I - B_2)| = |I - (B_1 \cup B_2)|$ by applying Case 1. Thus $|B_1 \cap B_2| = |B_1| + |B_2| - |B_1 \cup B_2| = 8 - 6 = 2$. Now $B_1 \triangle B_2$ is a c-block because $B_1 \triangle B_2$ consists of four elements in the Fano plane without containing 0 and a line in the Fano plane.

Case 3 (B_1 is an n-blockand B_2 is a n-block). Since $B_1 \cap B_2 \neq \emptyset, I - B_2$ and B_1 are two n-blocks in case 1. Thus $|B_1 \cap (I - B_2)| = 2$ and we obtain $|B_1 \cap B_2| = 2$. Moreover $B_1 \triangle B_2$ be an n-block since $|B_1 \triangle B_2| = 4$ and $B_1 \triangle B_2$ contains 0 and a line in the Fano plane. □

Remark. When $B_1 \cap B_2 = \emptyset$ (respectively, $B_1 = B_2$), $|B_1 \cap B_2| = 0$ (respectively, 4). Therefore for any two blocks B_1 and B_2 in $\mathcal{B}$, $|B_1 \cap B_2|$ must be an even number.

Now, we define swaps on I as follows.

Definition 6. A permutation $\sigma : I \to I$ is called a swap if it is an involution $\sigma^2 = 1$. We denote the set of all the swaps on I as S_w.

Remark.(1) A transposition (i,j) is the typical example of swaps. We call such swap σ a *simple swap* and denote it as $\sigma_{(i,j)}$.

(2) One can observe that each swap σ can be written as a finite composition of disjoint transpositions, which is uniquely determined up to permutation of transpositions.

(3) It is clear that S_w is a subset of the permutation group S_8, but not closed under the composition of bijections on I. Thus we perform the composition of swaps in S_w as elements in S_8.

We consider a swap $\sigma_{(i,j)} = (i,j)$ and call the image of a block of $\mathcal{B}$ via $\sigma_{(i,j)}$ an (i,j)-block. Here $\sigma_{(i,j)}(\mathcal{B})$ contains two new types of blocks in addition to n-blocks and c-blocks. Simply a block A in $\sigma_{(i,j)}(\mathcal{B})$ consisting of subscripts of a line and another vector (instead of e_0) in the Fano plane is called an α *-block*, and the complement of an α-block given as $I - A$ is called a β*-block*. Obviously, each β-block contains $0 \in I$.

For any subset A, B in I and a permutation $\tilde{\sigma}$ on I, one can show that

$$\tilde{\sigma}(A \triangle B) = \tilde{\sigma}(A) \triangle \tilde{\sigma}(B).$$

Therefore, we obtain the following lemma for (i,j)-blocks in $\sigma_{(i,j)}(\mathcal{B})$ by applying lemma 5.

Lemma 7. *For any two (i,j)-blocks B_1, B_2 in $\sigma_{(i,j)}(\mathcal{B})$, $|B_1 \cap B_2| = 2$ and $B_1 \triangle B_2$ is also an (i,j)-block unless $B_1 \cap B_2 = \emptyset$ or $B_1 = B_2$.*

3.2. *Integral subsets in octonions*

In this subsection, we define subsets in octonions $\mathbb{O}$ and show that they are integral subsets in octonions.

At first, we consider a subset $\mathbb{O}_{\mathcal{B}}$ in $\mathbb{O}$ by using the set of blocks $\mathcal{B}$ as

$$\mathbb{O}_{\mathcal{B}} := \left\{ \frac{1}{2}(\pm e_a \pm e_b \pm e_c \pm e_d) \in \mathbb{O} \,\middle|\, \{a,b,c,d\} \in \mathcal{B} \right\},$$

and we define a subset $\mathbb{O}_Z$ in $\mathbb{O}$ as

$$\mathbb{O}_Z := \mathrm{span}_{\mathbb{Z}} \mathbb{O}_{\mathcal{B}}.$$

Unfortunately, $\mathbb{O}_Z$ is not an integral subset in $\mathbb{O}$ because it is not closed under multiplication by checking

$$\frac{1}{2}(e_0 + e_1 + e_4 + e_5) \cdot \frac{1}{2}(e_4 + e_5 + e_6 + e_7) = \frac{1}{2}(-e_0 + e_3 + e_5 + e_7) \notin \mathbb{O}_Z.$$

Therefore, we apply simple swaps to produce new subsets in $\mathbb{O}$ from $\mathbb{O}_Z$ so that the new subset is closed under multiplication. Here we consider a simple swap $\sigma_{(i,j)}$ for each $\{i,j\} \subset I$ and

$$\mathbb{O}_{\mathcal{B}}(i,j) := \left\{ \frac{1}{2}(\pm e_a \pm e_b \pm e_c \pm e_d) \in \mathbb{O} \,\middle|\, \{a,b,c,d\} \in \sigma_{(i,j)}(\mathcal{B}) \right\}.$$

We also define a subset $\mathbb{O}_Z(i,j)$ in $\mathbb{O}$ as

$$\mathbb{O}_Z(i,j) := \mathrm{span}_{\mathbb{Z}}\mathbb{O}_{\mathcal{B}}(i,j).$$

In the following theorem, we have another way to express the elements in $\mathbb{O}_Z(i,j)$.

Theorem 8. *For each $\{i,j\} \subset I$ and the subset $\mathbb{O}_Z(i,j)$ in $\mathbb{O}$, we have*

$$\mathbb{O}_Z(i,j) = \left\{ \sum_{i=0}^{7} a_i e_i \in \mathbb{O} \;\middle|\; \begin{array}{r} \textit{simultaneously } a_i \in \mathbb{Z} \\ \textit{or simultaneously } a_i \in \mathbb{Z}+\frac{1}{2} \\ \textit{or only four } a_i \textit{ are in } \mathbb{Z}+\frac{1}{2} (\textit{the others are in } \mathbb{Z}) \\ \textit{whose choices are given from } \sigma_{(i,j)}(\mathcal{B}) \end{array} \right\}.$$

Note. We denote the set in the right-hand side as $S(i,j)$ and group the elements in $S(i,j)$ by three types (a) simultaneously $a_i \in \mathbb{Z}$, (b) simultaneously $a_i \in \mathbb{Z}+\frac{1}{2}$, and (c) only four a_i are in $\mathbb{Z}+\frac{1}{2}$ where the four choices are given by an (i,j)-block in $\sigma_{(i,j)}(\mathcal{B})$. The elements of type (a) are called Gravesian integers, and octonions whose coefficients are in either $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ are called Kleinian integers [2].

Proof of Theorem 8. (1) First we show that $\mathbb{O}_Z(i,j) \supset S(i,j)$.

For each $a \in I$, there exists an (i,j)-block $\{a,b,c,d\}$ in $\sigma_{(i,j)}(\mathcal{B})$ and we have

$$e_a = \frac{1}{2}(e_a + e_b + e_c + e_d) + \frac{1}{2}(e_a - e_b - e_c - e_d) \in \mathbb{O}_Z(i,j).$$

Moreover, since the linear combinations of e_a over $\mathbb{Z}$ are in $\mathbb{O}_Z(i,j)$, the elements of $S(i,j)$ of type (a) are in $\mathbb{O}_Z(i,j)$.

Each element x of $S(i,j)$ of type (b) is also in $\mathbb{O}_Z(i,j)$ because it is the linear combination of $\frac{1}{2}\sum_{i=0}^{7} e_i$ in $\mathbb{O}_Z(i,j)$ and $x - \frac{1}{2}\sum_{i=0}^{7} e_i$ which is an element of $S(i,j)$ of type (a).

For an element x of $S(i,j)$ of type (c), we obtain $\frac{1}{2}(e_a + e_b + e_c + e_d) \in \mathbb{O}_Z(i,j)$ from the corresponding four choices given by (i,j)-block $\{a,b,c,d\}$ in $\sigma_{(i,j)}(\mathcal{B})$. Moreover since $x - \frac{1}{2}(e_a + e_b + e_c + e_d)$ is an element of $S(i,j)$ of type (a), x is in $\mathbb{O}_Z(i,j)$.

Thus we show that $\mathbb{O}_Z(i,j) \supset S(i,j)$.

(2) we show that $\mathbb{O}_Z(i,j) \subset S(i,j)$.

Because it is clear that $\sigma_{(i,j)}(\mathbb{O}_{\mathcal{B}}(i,j)) \subset S(i,j)$ by definition of $S(i,j)$, it is enough to show that $S(i,j)$ is closed under linear combinations over $\mathbb{Z}$. To show this, we only need to check the sum of two elements in $S(i,j)$ of type (c). Consider x and y in $S(i,j)$ of type (c) corresponded to (i,j)-block B_x and B_y. Here $|B_x \cap B_y|$ can be $0, 2, 4$ by Lemma 7. When $|B_x \cap B_y| = 0$ (respectively, 4), we have $B_x \cap B_y = \emptyset$ (respectively, $B_x = B_y$), and $x + y$ is an element in $S(i,j)$ of type (b) (respectively, (a)). When $|B_x \cap B_y| = 2$, $B_1 \triangle B_2$ is also an (i,j)-block by Lemma 7, and $x + y$ is an element in $S(i,j)$ of type (c).

This shows that $\mathbb{O}_Z(i,j) \subset S(i,j)$. □

Remark. Since the subset of all the type (a) forms an lattice

$$\mathbb{O}_0 := \mathrm{span}_{\mathbb{Z}} \left\{ \sum_{i=0}^{7} a_i e_i \in \mathbb{O} \;\middle|\; a_i \in \mathbb{Z} \right\},$$

we have $\mathbb{O}_0 \subset \mathbb{O}_Z(i,j)$ by Theorem 8. Therefrom, we define an equivalent relationship

$$x \sim y \quad x, y \in \mathbb{O}_Z(i,j) :\Longleftrightarrow x - y \;\in \mathbb{O}_0.$$

We also denote it by $x \equiv y \mod \mathbb{O}_0$. By Theorem 8 the set of equivalence classes is bijective to $\{0, \frac{1}{2}\sum_{i=0}^{7} e_i\} \cup \mathbb{O}_{\mathcal{B}}(i,j)$.

Integral conditions for $O_Z(i,j)$ in O

Now, we check the conditions of integral subsets $\mathbb{O}_Z(i,j)$ in $\mathbb{O}$.

(a) By applying Theorem 8, each $x = a_0e_0 + a_1e_1 + \cdots + a_7e_7$ in $\mathbb{O}_Z(i,j)$ is one of three types in $S(i,j)$. For each type, we have $2(x,1) = 2(x,e_0) = 2a_0 \in \mathbb{Z}$. For type (c) with corresponding (i,j)-block $\{a,b,c,d\}$ in $\sigma_{(i,j)}(\mathcal{B})$

$$\begin{aligned}\sum_{i=0}^{7} a_i^2 &= \sum_{i\in I-\{a,b,c,d\}} a_i^2 + \sum_{i\in\{a,b,c,d\}} \left(\left(a_i - \frac{1}{2}\right) + \frac{1}{2}\right)^2 \\ &= \sum_{i\in I-\{a,b,c,d\}} a_i^2 + \sum_{i\in\{a,b,c,d\}} \left(a_i - \frac{1}{2}\right)^2 + \sum_{i\in\{a,b,c,d\}} \left(a_i - \frac{1}{2}\right) + 1 \in \mathbb{Z}.\end{aligned}$$

Similarly, for other types, we have $\sum_{i=0}^{7} a_i^2 \in \mathbb{Z}$. This shows condition (1) for integral subsets.

(b) Conditions (2) and (4) are trivial by definition of $\mathbb{O}_Z(i,j)$.

(c) Condition (5) can be checked by following the argument in Example 2. It is enough to show that for any x in $\mathbb{O}$, there is y in $\mathbb{O}_Z(i,j)$ with $\|x - y\| < 1$. According to remark of Theorem 8, we may consider $x = \sum_{i=0}^{7} a_i e_i$ where $a_i \in [0,1)$ for each i. We set cases according to the coefficients, then find y in $\mathbb{O}_Z(i,j)$ with $\|x - y\| < 1$. Since these cases can be checked by rather simple calculations, we skip the full process. One special case is $x = \sum_{i=0}^{7} a_i e_i$ where $a_i \in \{\frac{1}{4}, \frac{3}{4}\}$ for each i. We choose $y = \sum_{i=0}^{7} \frac{1}{2} e_i$ in $\mathbb{O}_Z(i,j)$, and obtain $\|x - y\| \le 8(\frac{1}{4})^2 < 1$.

(d) Condition (3) for the multiplication on $\mathbb{O}_Z(i,j)$ is not trivial. In the following subsection, we show that $\mathbb{O}_Z(i,j)$ is closed under multiplication.

The above argument is summarized as the following main theorem.

Theorem 9. *For each $\{i,j\} \subset I$, the subset $\mathbb{O}_Z(i,j)$ in $\mathbb{O}$ is an integral subset in $\mathbb{O}$.*

3.3. *Multiplication on* $\mathbb{O}_\mathbf{Z}(\mathbf{i},\mathbf{j})$

In this subsection, we show $\mathbb{O}_Z(i,j)$ for each $\{i,j\} \subset I$ is closed under multiplication. To do that, we show: (1) it is enough to show a special case of $\{i,j\} \subset I$, and (2) $\mathbb{O}_Z(0,7)$ is closed under multiplication.

M-(1) It is enough to show a special case of $\{0,7\} \subset I$

We separate $\{i,j\} \subset I$ into two cases (a) $\{0,j\}$, and (b) $\{i,j\}$ where $i,j \neq 0$.

The multiplication issue for the cases of $\mathbb{O}_Z(0,j)$ is clearly equivalent to $\mathbb{O}_Z(0,7)$ because of the symmetry on Fano plane.

For the cases of $\{i,j\}$, there is a unique $k \in I$ so that $\{0,i,j,k\}$ is an n-block in $\mathcal{B}$. We claim $\mathbb{O}_Z(i,j) = \mathbb{O}_Z(0,k)$. Therefrom, $\mathbb{O}_Z(0,k)$ is closed under multiplication, $\mathbb{O}_Z(i,j)$ is closed under multiplication and therefore we prove the claim in the following proposition.

Proposition 10. *For any n-block* $\{0,i,j,k\}$ *in* $\mathcal{B}, \mathbb{O}_Z(i,j) = \mathbb{O}_Z(0,k)$.

Proof. We consider a permutation $\sigma_{(i,j)}\sigma_{(0,k)} : I \to I$ and show $\sigma_{(i,j)}\sigma_{(0,k)}$ preserves the set of blocks $\mathcal{B}$.

Consider an n-block B in $\mathcal{B}$, the n-block B is either $\{0,i,j,k\}$ itself or another n-block whose intersection with $\{0,i,j,k\}$ is one of $\{0,i\}$, $\{0,j\}$, $\{0,k\}$ (by Lemma 5). If B is $\{0,i,j,k\}$ or another n-block whose intersection with $\{0,i,j,k\}$ is $\{0,k\}$, clearly $\sigma_{(i,j)}\sigma_{(0,k)}(B) = B$. If B is another n-block whose intersection with $\{0,i,j,k\}$ is $\{0,i\}$ (respectively, $\{0,j\}$), $\sigma_{(i,j)}\sigma_{(0,k)}(B)$ is a c-block whose intersection with $\{0,i,j,k\}$ is $\{k,j\}$ (respectively, $\{k,i\}$). For a c-block C in $\mathcal{B}$, the $I - C$ is an n-block in $\mathcal{B}$. By the above, $\sigma_{(i,j)}\sigma_{(0,k)}(I - C) = I - \sigma_{(i,j)}\sigma_{(0,k)}(C)$ is a block in $\mathcal{B}$, and $\sigma_{(i,j)}\sigma_{(0,k)}(C)$ is also a block in $\mathcal{B}$.

Since $\sigma_{(i,j)}\sigma_{(0,k)}$ preserves the set of blocks $\mathcal{B}$, we get $\sigma_{(i,j)}(\mathcal{B}) = \sigma_{(0,k)}(\mathcal{B})$ and obtain

$$\mathbb{O}_Z(i,j) = \mathrm{span}_{\mathbb{Z}}(\mathbb{O}_{\mathcal{B}}(i,j)) = \mathrm{span}_{\mathbb{Z}}(\mathbb{O}_{\mathcal{B}}(0,k)) = \mathbb{O}_Z(0,k). \qquad \square$$

In conclusion, in order to show that $\mathbb{O}_Z(i,j)$ for each $\{i,j\} \subset I$ is closed under multiplication, it is enough to check a special case of $\{i,j\} \subset I$. Thus we work on $\mathbb{O}_Z(0,7)$ below.

M-(2) $\mathbb{O}_\mathbf{Z}(\mathbf{0},\mathbf{7})$ is closed under multiplication

(1) Transformations on $(\mathbf{0},\mathbf{7})$-blocks:

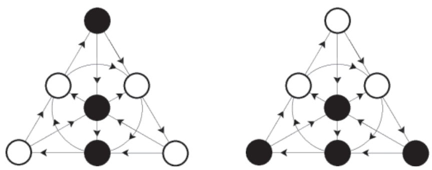
Before we show that $\mathbb{O}_Z(0,7)$ is closed under multiplication, we perform a series of transformations on $(0,7)$-blocks given by the multiplication of each e_i to obtain the configuration of $(0,7)$-blocks. Here we use a diagram derived from Fano plane to present the transformations.

We group the $(0,7)$-blocks in $\sigma_{(0,7)}(\mathcal{B})$ by a type A block ($\in \sigma_{(0,7)}(\mathcal{B}) - \mathcal{B}$) and a type B block ($\in \sigma_{(0,7)}(\mathcal{B}) \cap \mathcal{B}$). Here $\sigma_{(0,7)}$ acts on $\sigma_{(0,7)}(\mathcal{B}) \cap \mathcal{B}$ but it does not preserve each element in $\sigma_{(0,7)}(\mathcal{B}) \cap \mathcal{B}$. Type A blocks consist of α-blocks and β-blocks. In below, we list the $(0,7)$-blocks by type:

Type A	α-block	$\{7,1,2,3\}\{7,1,4,5\}\{7,2,4,6\}\{7,3,5,6\}$
	β-block	$\{4,5,6,0\}\{2,3,6,0\}\{1,3,5,0\}\{1,2,4,0\}$
Type B	n-block	$\{0,3,4,7\}\{0,2,5,7\}\{0,1,6,7\}$
	c-block	$\{1,2,5,6\}\{1,3,4,6\}\{2,3,4,5\}$

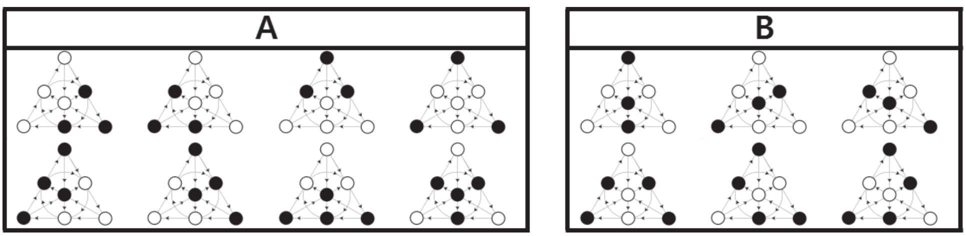
(0,7)-blocks

Moreover, we introduce a diagram presenting blocks using Fano plane. For each block, we mark vectors in the Fano plane according to the subscripts in the box. For example, the diagrams of n-block $\{0,3,4,7\}$ and α-block $\{3,5,6,7\}$ are presented as below:



{0,3,4,7} {3,5,6,7}

The diagrams of blocks in $(0,7)$-blocks are listed below:



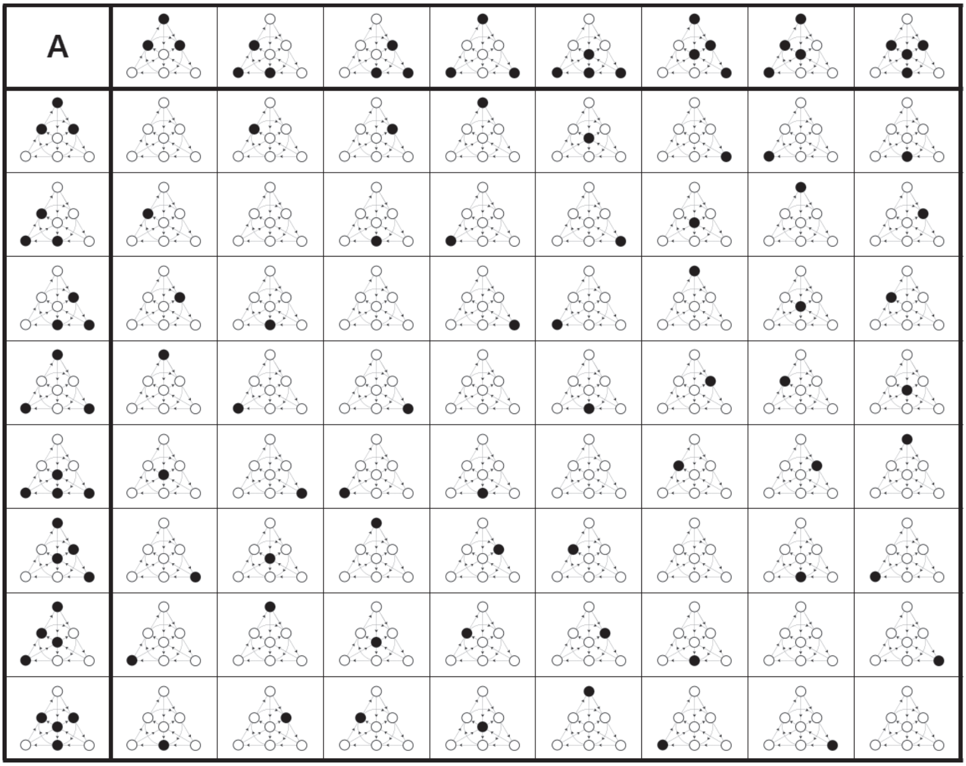
Diagrams of (0,7)-blocks

Now we define permutation τ_i on I defined for e_i as

$$\begin{aligned} \tau_i &: I \to I, \\ & j \mapsto \tau_i(j) := \text{subscript of } e_i \cdot e_j. \end{aligned}$$

Note that when we define $\tau_i(j)$, we ignore the signature of $e_i \cdot e_j$. Moreover, the permutation τ_i induces a transformation on the (i,j)-blocks, which is also denoted τ_i. The transformations involve tedious calculations of octonions. In below we use block diagrams to present the transformation of τ_i on type A blocks in

$(0, 7)$-blocks:



Transformation of type A blocks in (0,7)-blocks

In the above table, the transformation $\tau_i (i \neq 0)$ is presented by the Fano plane with a mark on e_i, and the transformation τ_0 is also presented by the Fano plane with no mark. The table presents that a type A block in the kth row in the left end column is sent to the type A block in the lth column in the top row via the transformation presented by the diagram in (k, l) coordinates.

From the above table, we obtain the following useful lemma.

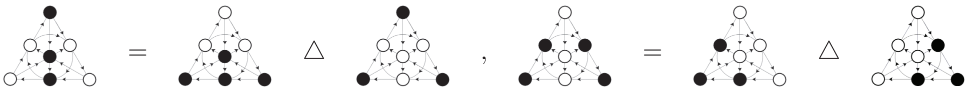
Lemma 11. (1) *For each $i \in I$ and corresponding transformation τ_i, the set of type A blocks in $(0, 7)$-blocks is preserved by τ_i.*

(2) *For each pair of type A $(0, 7)$-blocks A_1 and A_2, there is $i \in I$ such that $\tau_i(A_1) = A_2$ and $\tau_i(A_2) = A_1$.*

(3) *For each type B $(0, 7)$-block B_1, there is a pair of type A $(0, 7)$-blocks A_1 and A_2 such that $B_1 = A_1 \triangle A_2$.*

Proof. (1) and (2) are clear from the above table.

(3) Thanks to the symmetry of the Fano plane, we only need to check the following two cases:



This proves the lemma. □

(2) Multiplication on $O_Z(0,7)$:

We show the multiplication on $\mathbb{O}_Z(0,7)$ by checking the following steps.

(a) For each element x in $\mathbb{O}_Z(0,7)$ given by a type A block in $\sigma_{(0,7)}(\mathcal{B})$ and e_i $(i \in I)$, $x \cdot e_i$ and $e_i \cdot x$ are also elements in $\mathbb{O}_Z(0,7)$ given by type A blocks.
(b) For each pair of x and y in $\mathbb{O}_Z(0,7)$ given by two type A blocks in $\sigma_{(0,7)}(\mathcal{B})$, there is a pair of x' and y' in $\mathbb{O}_{\mathcal{B}}(0,7) \subset \mathbb{O}_Z(0,7)$ with $x \equiv x'$, $y \equiv y' \mod \mathbb{O}_0$ satisfying $x' \cdot y' \in \mathbb{O}_Z(0,7)$.
(c) For each pair of x and y in $\mathbb{O}_Z(0,7)$ given by two type A blocks in $\sigma_{(0,7)}(\mathcal{B})$, $x \cdot y \in \mathbb{O}_Z(0,7)$.
(d) $\mathbb{O}_Z(0,7)$ is closed under multiplication.

Note. We denote $\mathcal{D}$ as the subset of elements in $\mathbb{O}_Z(0,7)$ given by a type A block in $\sigma_{(0,7)}(\mathcal{B})$.

Step (a). For each element x in $\mathcal{D}$ and e_i $(i \in I)$, $x \cdot e_i$ and $e_i \cdot x$ are in $\mathcal{D}$.

Proof. Since $e_i e_j = -e_j e_i$ for $i \neq j$, it is enough to check $e_i \cdot x$ in $\mathbb{O}_Z(0,7)$ for each x in $\mathbb{O}_Z(0,7)$.

For each $x = \sum_{j=0}^{7} a_j e_j$, we have $e_j x = \sum_{j=0}^{7} \pm a_j e_{\tau_i(j)}$. Since x is type (c) given by type A, x can be written as $x = x' + \frac{1}{2}(e_a + e_b + e_c + e_d)$ where x' is type (a) and $\{a,b,c,d\}$ is a type A block in $\sigma_{(0,7)}(\mathcal{B})$, and moreover $e_j x = e_j x' + \frac{1}{2}(e_{\tau_i(a)} + e_{\tau_i(b)} + e_{\tau_i(c)} + e_{\tau_i(d)})$. Here $\{\tau_i(a), \tau_i(b), \tau_i(c), \tau_i(d)\}$ is also a type A block in $\sigma_{(0,7)}(\mathcal{B})$ by Lemma 11, and $e_j x$ is an element in $\mathbb{O}_Z(0,7)$ given by a type A block. □

Step (b). For each pair of x and y in $\mathcal{D}$, there is a pair of x' and y' in $\mathbb{O}_{\mathcal{B}}(0,7) \subset \mathbb{O}_Z(0,7)$ with $x \equiv x'$, $y \equiv y' \mod \mathbb{O}_0$ satisfying $x' \cdot y' \in \mathbb{O}_Z(0,7)$.

Proof. Let x and y be a pair of elements in $\mathbb{O}_Z(0,7)$ given by two type A blocks $A_x = \{a,b,c,d\}$ and A_y respectively. By Lemma 11, there is $j \in I$ such that $\tau_j(A_x) = A_y$. We can choose $x' = \frac{1}{2}(e_a + e_b + e_c + e_d)$ and take $y'' = \frac{1}{2}(e_{\tau_j(a)} + e_{\tau_j(b)} + e_{\tau_j(c)} + e_{\tau_j(d)})$ so that $x \equiv x'$, $y \equiv y'' \mod \mathbb{O}_0$. Moreover, we adjust the coefficients of y'' to get $y' = x' \cdot e_j$ according to $\mod \mathbb{O}_0$ so that y' still satisfies $y \equiv y' \equiv y'' \mod \mathbb{O}_0$. Now we have $x' \cdot y' = x' \cdot (x' \cdot e_j) = (x')^2 \cdot e_j$.

By applying the rank equation,

$$(x')^2 = 2(x',1)x' - 1 = \begin{cases} \pm x' - 1 & \text{if } (x',1) = \pm\frac{1}{2}, \\ -1 & \text{if } (x',1) = 0. \end{cases}$$

Therefrom, $(x')^2 \cdot e_j$ is in $\mathbb{O}_Z(0,7)$ by step (a). This shows step (b). □

Step (c). For each pair of x and y in $\mathcal{D}$, $x \cdot y \in \mathbb{O}_Z(0,7)$.

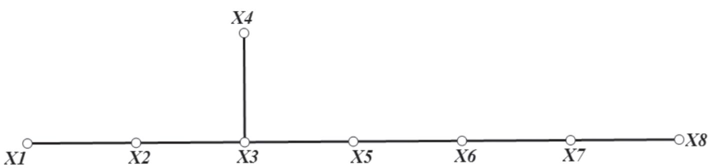
Proof. By step (b), there is a pair of x' and y' in $\mathbb{O}_{\mathcal{B}}(0,7) \subset \mathbb{O}_Z(0,7)$ with $x \equiv x'$, $y \equiv y' \mod \mathbb{O}_0$ satisfying $x' \cdot y' \in \mathbb{O}_Z(0,7)$. According to the remark of Theorem 8, x and y can be written as $x = x_0 + x'$ and $y = y_0 + y'$ where x_0 and y_0 are type (a). Here, (1) since $\mathbb{O}_0 = \text{span}_{\mathbb{Z}}\{\sum_{i=0}^{7} a_i e_i \in \mathbb{O} \mid a_i \in \mathbb{Z}\} \subset \mathbb{O}_Z(0,7)$ is clearly closed under multiplication, $x_0 \cdot y_0$ is in $\mathbb{O}_Z(0,7)$, (2) by step (a), $x_0 \cdot y'$ and $x' \cdot y_0$ are in $\mathbb{O}_Z(0,7)$, and (3) by step (b), $x' \cdot y'$ is in $\mathbb{O}_Z(0,7)$. Therefrom, we obtain $xy \in \mathbb{O}_Z(0,7)$. □

Step (d). $\mathbb{O}_Z(0,7)$ is closed under multiplication.

Proof. By Lemma 11, any element in $\mathbb{O}_Z(0,7)$ given by a type B block in $\sigma_{(0,7)}(\mathcal{B})$ can be expressed as a sum of two elements in $\mathcal{D}$. Thus we obtain $\mathbb{O}_Z(0,7) = \text{span}_{\mathbb{Z}}(\mathcal{D})$. Since the multiplication of any two elements in $\mathcal{D}$ is in $\mathbb{O}_Z(0,7)$ by step (c), $\mathbb{O}_Z(0,7)$ is closed under multiplication. □

3.4. *E_8 root lattice in $\mathbb{O}$*

Now we can identify the integral subset $\mathbb{O}_Z(0,7)$ as an E_8-lattice in $\mathbb{O}$. Here an E_8 lattice in $\mathbb{O}$ is generated by the elements in $\mathbb{O}_Z(0,7)$ with length 1 via the Weyl action. The action is given by the following Dynkin diagram of E_8:



E_8 Dynkin diagram

Here, X_i $i \in I$ in the diagrams are chosen as follows:

$$X_1 = \frac{1}{2}(e_0 - e_2 + e_3 - e_6), \quad X_5 = \frac{1}{2}(e_1 + e_2 + e_3 + e_7),$$

$$X_2 = \frac{1}{2}(-e_0 - e_4 + e_5 + e_6), \quad X_6 = \frac{1}{2}(-e_2 - e_3 + e_4 + e_5),$$

$$X_3 = \frac{1}{2}(-e_1 + e_4 - e_5 - e_7), \quad X_7 = \frac{1}{2}(e_0 - e_4 - e_5 + e_6),$$

$$X_4 = \frac{1}{2}(e_1 - e_3 - e_4 - e_6), \quad X_8 = \frac{1}{2}(-e_0 - e_1 - e_6 + e_7).$$

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